Theoretical Properties

- Structural Operational Semantics

- Correctness of Live Variables Analysis

The Semantics

A *state* is a mapping from variables to integers:

$$\sigma \in \mathbf{State} \;=\; \mathbf{Var} \to \mathbf{Z}$$

The semantics of arithmetic and boolean expressions

$$\mathcal{A}: \;\; \mathbf{AExp} \to (\mathbf{State} \to \mathbf{Z}) \quad \text{(no errors allowed)}$$

$$\mathcal{B}: \;\; \mathbf{BExp} \to (\mathbf{State} \to \mathbf{T}) \quad \text{(no errors allowed)}$$

The *transitions* of the semantics are of the form

$$\langle S, \sigma \rangle \to \sigma' \quad \text{and} \quad \langle S, \sigma \rangle \to \langle S', \sigma' \rangle$$

Transitions

$$\langle [x := a]^\ell, \sigma\rangle \rightarrow \sigma[x \mapsto \mathcal{A}[\![a]\!]\sigma]$$

$$\langle [\texttt{skip}]^\ell, \sigma\rangle \rightarrow \sigma$$

$$\frac{\langle S_1, \sigma\rangle \rightarrow \langle S_1', \sigma'\rangle}{\langle S_1; S_2, \sigma\rangle \rightarrow \langle S_1'; S_2, \sigma'\rangle}$$

$$\frac{\langle S_1, \sigma\rangle \rightarrow \sigma'}{\langle S_1; S_2, \sigma\rangle \rightarrow \langle S_2, \sigma'\rangle}$$

$$\langle \texttt{if } [b]^\ell \texttt{ then } S_1 \texttt{ else } S_2, \sigma\rangle \rightarrow \langle S_1, \sigma\rangle \qquad \text{if } \mathcal{B}[\![b]\!]\sigma = \textit{true}$$

$$\langle \texttt{if } [b]^\ell \texttt{ then } S_1 \texttt{ else } S_2, \sigma\rangle \rightarrow \langle S_2, \sigma\rangle \qquad \text{if } \mathcal{B}[\![b]\!]\sigma = \textit{false}$$

$$\langle \texttt{while } [b]^\ell \texttt{ do } S, \sigma\rangle \rightarrow \langle (S; \texttt{while } [b]^\ell \texttt{ do } S), \sigma\rangle \qquad \text{if } \mathcal{B}[\![b]\!]\sigma = \textit{true}$$

$$\langle \texttt{while } [b]^\ell \texttt{ do } S, \sigma\rangle \rightarrow \sigma \qquad \text{if } \mathcal{B}[\![b]\!]\sigma = \textit{false}$$

Example:

$$\langle [\texttt{y:=x}]^1; [\texttt{z:=1}]^2; \texttt{while } [\texttt{y>1}]^3 \texttt{ do } ([\texttt{z:=z*y}]^4; [\texttt{y:=y-1}]^5); [\texttt{y:=0}]^6, \sigma_{300}\rangle$$

$$\rightarrow \langle [\texttt{z:=1}]^2; \texttt{while } [\texttt{y>1}]^3 \texttt{ do } ([\texttt{z:=z*y}]^4; [\texttt{y:=y-1}]^5); [\texttt{y:=0}]^6, \sigma_{330}\rangle$$

$$\rightarrow \langle \texttt{while } [\texttt{y>1}]^3 \texttt{ do } ([\texttt{z:=z*y}]^4; [\texttt{y:=y-1}]^5); [\texttt{y:=0}]^6, \sigma_{331}\rangle$$

$$\rightarrow \langle [\texttt{z:=z*y}]^4; [\texttt{y:=y-1}]^5; \texttt{while } [\texttt{y>1}]^3 \texttt{ do } ([\texttt{z:=z*y}]^4; [\texttt{y:=y-1}]^5); [\texttt{y:=0}]^6, \sigma_{331}\rangle$$

$$\rightarrow \langle [\texttt{y:=y-1}]^5; \texttt{while } [\texttt{y>1}]^3 \texttt{ do } ([\texttt{z:=z*y}]^4; [\texttt{y:=y-1}]^5); [\texttt{y:=0}]^6, \sigma_{333}\rangle$$

$$\rightarrow \langle \texttt{while } [\texttt{y>1}]^3 \texttt{ do } ([\texttt{z:=z*y}]^4; [\texttt{y:=y-1}]^5); [\texttt{y:=0}]^6, \sigma_{323}\rangle$$

$$\rightarrow \langle [\texttt{z:=z*y}]^4; [\texttt{y:=y-1}]^5; \texttt{while } [\texttt{y>1}]^3 \texttt{ do } ([\texttt{z:=z*y}]^4; [\texttt{y:=y-1}]^5); [\texttt{y:=0}]^6, \sigma_{323}\rangle$$

$$\rightarrow \langle [\texttt{y:=y-1}]^5; \texttt{while } [\texttt{y>1}]^3 \texttt{ do } ([\texttt{z:=z*y}]^4; [\texttt{y:=y-1}]^5); [\texttt{y:=0}]^6, \sigma_{326}\rangle$$

$$\rightarrow \langle \texttt{while } [\texttt{y>1}]^3 \texttt{ do } ([\texttt{z:=z*y}]^4; [\texttt{y:=y-1}]^5); [\texttt{y:=0}]^6, \sigma_{316}\rangle$$

$$\rightarrow \langle [\texttt{y:=0}]^6, \sigma_{316}\rangle$$

$$\rightarrow \sigma_{306}$$

Equations and Constraints

Equation system $\mathsf{LV}^{=}(S_\star)$:

$$\mathsf{LV}_{exit}(\ell) = \begin{cases} \emptyset & \text{if } \ell \in \mathit{final}(S_\star) \\ \bigcup\{\mathsf{LV}_{entry}(\ell') \mid (\ell', \ell) \in \mathit{flow}^R(S_\star)\} & \text{otherwise} \end{cases}$$

$$\mathsf{LV}_{entry}(\ell) = (\mathsf{LV}_{exit}(\ell) \backslash \mathit{kill}_{\mathsf{LV}}(B^\ell)) \cup \mathit{gen}_{\mathsf{LV}}(B^\ell)$$

where $B^\ell \in \mathit{blocks}(S_\star)$

Constraint system $\mathsf{LV}^{\subseteq}(S_\star)$:

$$\mathsf{LV}_{exit}(\ell) \supseteq \begin{cases} \emptyset & \text{if } \ell \in \mathit{final}(S_\star) \\ \bigcup\{\mathsf{LV}_{entry}(\ell') \mid (\ell', \ell) \in \mathit{flow}^R(S_\star)\} & \text{otherwise} \end{cases}$$

$$\mathsf{LV}_{entry}(\ell) \supseteq (\mathsf{LV}_{exit}(\ell) \backslash \mathit{kill}_{\mathsf{LV}}(B^\ell)) \cup \mathit{gen}_{\mathsf{LV}}(B^\ell)$$

where $B^\ell \in \mathit{blocks}(S_\star)$

Lemma

Each solution to the equation system $\mathsf{LV}^{=}(S_\star)$ is also a solution to the constraint system $\mathsf{LV}^{\subseteq}(S_\star)$.

Proof: Trivial.

Lemma

The least solution to the equation system $\mathsf{LV}^{=}(S_\star)$ is also the least solution to the constraint system $\mathsf{LV}^{\subseteq}(S_\star)$.

Proof: Use Tarski's Theorem.

Naive Proof: Proceed by contradiction. Suppose some LHS is strictly greater than the RHS. Replace the LHS by the RHS in the solution. Argue that you still have a solution. This establishes the desired contradiction.

Lemma

A solution *live* to the constraint system is preserved during computation

$$
\begin{array}{ccccccccc}
\langle S, \sigma_1 \rangle & \to & \langle S', \sigma_1' \rangle & \to & \cdots & \to & \langle S'', \sigma_1'' \rangle & \to & \sigma_1''' \\
\updownarrow \models \mathsf{LV}^{\subseteq} & & \updownarrow \models \mathsf{LV}^{\subseteq} & & & & \updownarrow \models \mathsf{LV}^{\subseteq} & & \\
\textit{live} & & \textit{live} & & \cdots & & \textit{live} & &
\end{array}
$$

Proof: requires a lot of machinery — see the book.

Correctness Relation

$$\sigma_1 \sim_V \sigma_2$$

means that for all practical purposes the two states σ_1 and σ_2 are equal: only the values of the live variables of V matters and here the two states are equal.

Example:

Consider the statement $[\mathtt{x:=y+z}]^\ell$

Let $V_1 = \{\mathtt{y}, \mathtt{z}\}$. Then $\sigma_1 \sim_{V_1} \sigma_2$ means $\sigma_1(\mathtt{y}) = \sigma_2(\mathtt{y}) \wedge \sigma_1(\mathtt{z}) = \sigma_2(\mathtt{z})$

Let $V_2 = \{\mathtt{x}\}$. Then $\sigma_1 \sim_{V_2} \sigma_2$ means $\sigma_1(\mathtt{x}) = \sigma_2(\mathtt{x})$

Correctness Theorem

The relation "$\sim$" is *invariant* under computation: the live variables for the initial configuration remain live throughout the computation.

$$
\begin{array}{ccccccccc}
\langle S, \sigma_1 \rangle & \to & \langle S', \sigma_1' \rangle & \to & \cdots & \to & \langle S'', \sigma_1'' \rangle & \to & \sigma_1''' \\
\updownarrow \sim_V & & \updownarrow \sim_{V'} & & & & \updownarrow \sim_{V''} & & \updownarrow \sim_{V'''} \\
\langle S, \sigma_2 \rangle & \to & \langle S', \sigma_2' \rangle & \to & \cdots & \to & \langle S'', \sigma_2'' \rangle & \to & \sigma_2'''
\end{array}
$$

$V = \mathit{live}_{entry}(\mathit{init}(S))$

$V'' = \mathit{live}_{entry}(\mathit{init}(S''))$

$V' = \mathit{live}_{entry}(\mathit{init}(S'))$

$$
\begin{aligned}
V''' &= \mathit{live}_{exit}(\mathit{init}(S'')) \\
&= \mathit{live}_{exit}(\ell)
\end{aligned}
$$

for some $\ell \in \mathit{final}(S)$